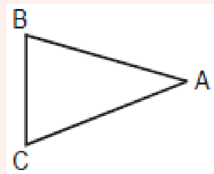
30 Important Questions

Direction for questions 1 and 2: Answer the questions based on the following information.

A cow is tethered at point A by a rope. Neither the rope nor the cow is allowed to enter ΔABC.



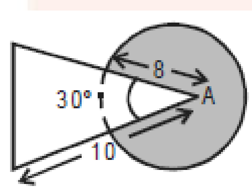
$\angle BAC = 30^\circ$
$l(AB) = l(AC) = 10\,m$

1. What is the area that can be grazed by the cow if the length of the rope is 8 m?

(A) $134\pi\frac{1}{3}$ sq.m (B) 121 π sq.m

(C) 132π sq.m (D) $\frac{176\pi}{3}$ sq.m

@ [D]



It can be seen that if the length of the rope is 8 m, then the cow will be able to graze an area equal to (the area of the circle with radius 8m) – (Area of the sector of the same circle with angle 30°) = $\pi(8)^2 - \frac{30}{360}\pi(8)^2 = 64\pi - \frac{1}{12}(64\pi)$

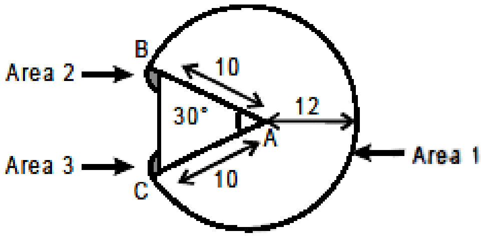
$= 64\pi\left(\frac{11}{12}\right) = \frac{176\pi}{3}$ sq.m

2. What is the area that can be grazed by the cow if the length of the rope is 12 m?

(A) $134\pi\frac{1}{3}$ sq.m (B) 121 π sq.m

(C) 132π sq.m (D) $\frac{176\pi}{3}$ sq.m

@ [A]



If the length of the rope is 12 m, then the total area thatcan be grazed by the cow is as depicted in the diagram.

Area 1 is (the area of the circle with radius 12) –(Area of the sector of the same circle with angle 30º)

So area 1 $= \pi(12)^2 - \frac{30}{360}\pi(12)^2 = 132\pi$

Since the length of the rope is higher than the sides ofthe triangle (viz. AB and AC), if the cow reaches pointB or C, there would still be a part of the rope (12 – 10)= 2 m in length. With this extra length available the cowcan further graze an area equivalent to some part ofthe circle with radius = 2 m from both points, i.e. B andC. This is depicted as area 2 and area 3 in the diagram.

Hence, the actual area grazed will be slightly morethan 132π. The only answer choice that supportsthis is (A).

3. A yearly payment to the servant is Rs. 90 plus one turban. The servant leaves the job after 9 months and receives Rs. 65 and a turban. Then find the price of the turban.

(A) Rs. 10

(B) Rs. 15

(C) Rs. 7.50

(D) Cannot be determined

@ [A]

Let the cost of the turban be T.

Total payment for one year = Rs. 90 + T. So the payment for 9 months should be 3/4 (90 + T). But this is equal to (65 + T). Equating the two, we get T = Rs. 10.

4. Which of the following values of x do not satisfy the inequality ($x^2 - 3x + 2 > 0$) at all?

(A) $1 \le x \le 2$

(B) $-1 \ge x \ge -2$

(C) $0 \le x \le 2$

(D) $0 \ge x \ge -2$

@ [A]

If we simplify the expression x2 – 3x + 2 > 0, we get(x – 1)(x – 2) > 0. For this product to be greater thanzero, either both the factors should be greater than zeroor both of them should be less than zero. Therefore,(x – 1) > 0 and (x – 2) > 0 or (x – 1) < 0 and (x – 2) < 0.

Hence, x > 1 and x > 2 or x < 1 and x < 2. If we were toclub the ranges, we would get either x > 2 or x < 1. Sofor any value of x equal to or between 1 and 2, theabove equation does not follow.

5. Given the quadratic equation $x^2 – (A – 3)x – (A – 2)$, for what value of A will the sum of the squares of theroots be zero?

(A) –2 (B) 3

(C) 6 (D) None of these

@ [D]

6. A man travels from A to B at a speed x km/hr. He then rests at B for x hours. He then travels from B to C at a speed 2x km/hr and rests for 2x hours. He moves further to D at a speed twice as that between B and C. He thus reaches D in 16 hr. If distances A-B, B-C and C-D are all equal to 12 km, the time for which he rested at B could be

(A) 3 hr (B) 6 hr

(C) 2 hr (D) 4 hr

@ [A]

7. Out of two-thirds of the total number of basketball matches, a team has won 17 matches and lost 3 of them. What is the maximum number of matches that the team can lose and still win more than three fourths of the total number of matches, if it is true that no match can end in a tie?

(A) 4 (B) 6

(C) 5 (D) 3

@ [A]

The team has played a total of (17 + 3) = 20 matches. This constitutes 3/2 of the matches. Hence, total number of matches played = 30. To win 3/4 of them, a team has to win 22.5, i.e. at least win 23 of them. In other words, the team has to win a minimum of 6 matches (since it has already won 17) out of remaining 10. So it can lose a maximum of 4 of them.

8. Find the largest +ve integer n such that n^3+100 is divisible by n+10

(A) 900 (B) 890

(C) 870 (D) 910

@ [B]

We can write $n+10 \equiv 0(\text{mod}(n+10))$

$n \equiv -10(\text{mod}(n+10))$,

$n^3 \equiv (-10)^3(\text{mod}(n+10))$

$n^3+100 \equiv (-1000+100)(\text{mod}(n+10))$

$\equiv -900(\text{mod}(n+10))$

i.e., n+10 should divide -900. The largest such N is 900-10=890. As n+10 cannot be greater than $|-900| = 900$ and the greatest devisor of $|-900|$ is 900

So the largest +ve integer n, such that n^3+100 is divisible by n+10 is 890.

9. Find the sum of the digites in $2^{2000} \,.\, 5^{2004}$

(A) 13 (B) 10

(C) 25 (D) 17

@ [A]

$2^{2000} .5^{2004}$

$= 5^4.2^{2000}.5^{2000} = 625.\ 10^{2000}$

Therefore, Sum of digits =6+2+5=13

10. Find the last two (ten's and unit's) digit of $(2003)^{2003}$

(A) 33 (B) 27

(C) 36 (D) 49

@ [B]

Last two digits is remainder when number is divided by 100

$(2003)^2 = 3^2$ (mod 100) =9 (mod 100)

$(2003)^4 = 9^2$ (mod 100) = -19 (mod 100)

$(2003)^8 = (-19)^2$ (mod100) =61 (mod100)

$(2003)^{16} = 61^2$ (mod100) = 21(mod 100)

$(2003)^{32} = 21^2$ (mod100) = 41 (mod 100)

$(2003)^{40}=(2003)^{32}.(2003)^8$=41.61(mod 100)

$(2003)^{2000}=(2003^{40})^{50}=1^{50}$(mod100) =1(mod100)

$(2003)^{2003}=2003^{2000}.2003^2.2003^1$(mod100) =1.9.3(mod100) =27(mod100)

Last two digits of 20032003 =27

11. Find two numbers both lying between 60and 70, each of which divides 2^{48}-1

(A) 63 and 65 (B) 64 and 65

(C) 60 and 65 (D) None of these

@ [A]

$2^{48}-1=(2^6-1)(2^6+1)(2^{12}+1)(2^{24}+1)$

$2^{12}+1$ and $2^{24}+1$ are greater than 70. Therefore, Numbers between 60 and 70 are 2^6-1 and 2^6+1

i.e. 63 and 65

12. A printer numbers the pages of a book starting with 1. He uses 3189 digits in all. How many pages does the book have?

(A) 1074 (B) 1076

(C) 1078 (D) 1080

@ [A]

No. of digits used in 1 digit number = 9×1 = 9

No.of digits used in 2 digit number = 90×2 = 180

No. of digits used in 3 digit number = 900×3 = 2700

No.digits used till three digit numbers = 9 + 180 + 2700 = 2889

Remaining digits used for 4 digit numbers = 3189 – 2889 = 300

Therefore, number of 4 digit numbers = 300/4 = 75

Number of pages = 1074

13. Find the largest prime factor of $3^{12} + 2^{12} - 2.6^6$

(A) 19 (B) 21

(C) 23 (D) 25

@ [A]

$3^{12} + 2^{12} - 2.6^6$

$= (3^6)^2 + (2^6)^2 - 2.3^6.2^6$

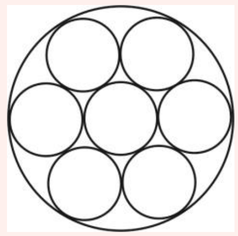
$= (3^6 - 2^6)^2 = \{(3^3 - 2^3)(3^3 + 2^3)\}^2$

$= \{(3-2)(3^2+3.2+2^2).(3+2)(3^2-3.2+2^2)$

$= \{19.5.7\}^2$

Therefore, Largest Prime Factor = 19

14. Seven cookles of radius 1 inch are cut from a circle of cookle dough, as shown, Neighboring cookes are tangent, and all except the center cookle are tangent to the edge of the dough. The leftover scrap is reshaped to form another cookle of the same thickness. What is the radius in inches of the scrap cookle?



(A) $\sqrt{2}$ (B) 1.5

(C) $\sqrt{\pi}$ (D) $\sqrt{2\pi}$

@ [A]

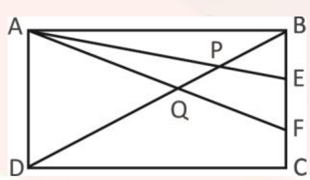
The big cookle has radius 3, since the center of the center cookle is the same as that of the large cookle. The difference in areas of the big cookle and the seven small ones is 2π. The scrap cookle has this area, so its radius must be $\sqrt{2}$.

15. In rectangle ABCD, AB = 6 and BC = 3 Point E between B and C, and point F between E and C are that BE = EF = FC. Segments $\overline{AE}$ and $\overline{AF}$ intersect $\overline{BD}$ at P and Q, respectively. The ratio BP : PQ : QD can be written as r : s : t where the greatest common factor or r, s and t is 1. What is r + s + t?

(A) 20 (B) 15

(C) 12 (D) 9

@ [A]



Since $\Delta APD \sim \Delta EPB$, $\frac{DP}{PB} = \frac{AD}{BE} = 3$, similarly $\frac{DQ}{QB} = \frac{3}{2}$. This means that $DQ = \frac{3 \cdot BD}{5}$. As ΔADP and ΔBEP are similar, we see that $\frac{PD}{PB} = \frac{3}{1}$. Thus $PB = \frac{BD}{4}$.

Therefore $r : s : t = \frac{1}{4} : \frac{2}{5} - \frac{1}{4} : \frac{3}{5} = 5 : 2 : 12$, so r + s + t = 20

16. A rectangle with positive integer side lengths in cm has area A cm^2 and perimeter P cm. Which of the following numbers cannot equal A + P?

(A) 100 (B) 102

(C) 106 (D) 108

@ [B]

Let the rectangle's length be a and its width be b. Its area is ab and the perimeter is 2a + 2b.

Then A + P = ab + 2a + 2b. Factoring, we have (a + 2) (b + 2) – 4.

The only one of the answer choices that cannot be expressed in this form is 102, as 102 + 4 is twice a prime.

There would then be no way to express 106 as (a + 2)(b + 2), keeping a and b as positive integers.

17. For some positive integers p, there is a quadrilateral ABCD with positive integer side lengths, perimeter p, right angles at B and C, AB = 2 and CD = AD. How many different values of p < 2015 are possible?

(A) 30 (B) 31

(C) 61 (D) 63

@ [B]

Let BC = x and CD = AD = y be positive integers. Drop a perpendicular from A to CD to show that, using the Pythagorean Theorem, that

$x^2 + (x – 2)^2 = y^2$

Simplifying yields $x^2 – 4y + 4 = 0$, so $x^2 = 4(y – 1)$. Thus, y is one more than a perfect square.

The perimeter $p = 2 + x + 2y = 2y + 2\sqrt{y-1} + 2$ must be less than 2015. Simple calculations demonstrate that $y = 31^2 + 1 = 962$ is valid, but $y = 32^2 + 1 = 1025$ is not. On the lower side, y = 1 does not work (because x > 0), but $y = 1^2 + 1$ does work. Hence, there are 31 valid y (All y such that $y = n^2 + 1$ for $1 \le n \le 31$), and so our answer is 31.

18. Two sides of a triangle have lengths 10 and 15. The third side is the average of the lengths of the altitudes to the given sides. How long is the third side?

(A) 6 (B) 8

(C) 9 (D) 12

@ [D]

The shortest side length has the longest altitude perpendicular to it. The average of the two altitudes of given will be between the lengths of the two altitudes, therefore the length of the side perpendicular to that altitude will be between 10 and 15. The only answer choice that meets this requirement is.

19. All 20 diagonals are drawn in regular octagon, At how many distinct points in the interior of the octagon (not on the boundary) do two or more diagonals intersect?

(A) 49 (B) 65

(C) 70 (D) 96

@ [A]

Let the number of intersection be x. We know that $x \le \binom{8}{4} = 70$, as every 4 points forms a quadrilateral with intersecting diagonals. However, four diagonals intersect in the center, so need to subtract $\binom{4}{2} - 1 = 5$ from this count 70 – 5 = 65. Note that diagonals like $\overline{AD}$, $\overline{CG}$, and $\overline{BE}$ all intersect at the same point. There are 8 of this type with three diagonals intersecting at the same point, so we need to subtract 2 of the $\binom{3}{2}$ (one is kept as the actual the actual intersection). In the end, we obtain 65 – 16 = 49

20. The sums of three whole numbers taken in pairs are 12, 18, and 19, what is the middle number?

(A) 4 (B) 5

(C) 6 (D) 7

@ [D]

Let the three numbers be equal to a, b and c, We can now write three equations:

a + b = 12

b + c = 17

a + c = 19

adding these equations together, we get that

2(a + b + c) = 48 and

a + b + c = 24

Substituting the original equations into this one, we find

c + 12 = 24

a + 17 = 24

b + 19 = 24

Therefore, our numbers are 12, 7, and 5. The middle number is 7.

21. A pair of six-sided dice are labeled so that die has only even numbers (two each of 2, 4 and 6), and the other die has only odd number (two of each 1, 3, and 5). The pair of dice is rolled. What is the probability that the sum of the numbers on the tops of the two dice is 7?

(A) $\frac{1}{6}$ (B) $\frac{1}{5}$

(C) $\frac{1}{2}$ (D) $\frac{1}{3}$

@ [D]

Assume we roll the die with only first. For whatever value rolled, there are exactly 2 faces on the odd die that makes the sum 7. The odd die has 6 taces, so our probability is $\frac{1}{3}$.

22. Mary divides a circle into 12 sectors. The central angles of these sectors. measured in degrees. are all integers and they form an arithmetic sequence. What is the degree measure of the smallest possible sector angle?

(A) 5 (B) 8

(C) 10 (D) 12

@ [B]

$$\frac{a+a+11r}{2}\cdot 12 = 360$$

$$2a+11r = 60$$

$$a = \frac{60-11r}{2}$$

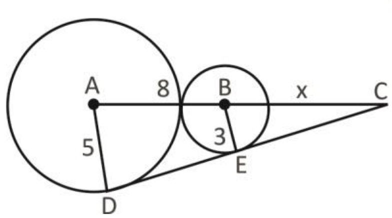
All sector angles are integer so r must be a multiple of 2. Plug in even integers for r starting from 2 to minimize a. We find this value to be 4 and the minimum value of a to be $\frac{60-11(4)}{2} = 8$

23. Externally tangent circles with centers at points A and B have radii of lengths 5 and 3 respectively. A line externally tangent both circles intersect ray AB at point C. What is BC?

(A) 4 (B) 4.8

(C) 10.2 (D) 12

@ [D]



Let D and E be the points of tangency on circles A and B with line CD.AB = 8. Also, let BC = 8, let BC = x. As $\angle ADC$ and $\angle BEC$ are right angles (a radius is perpendicular to a tangent line at the point to tangency) and both triangles share $\angle ACD$. $\Delta ADC \sim \Delta BEC$. From this we can get proportion.

24. A3 × 3 square is partitioned into 9 unit squares. Each square is painted either white or black with each color being likely, chosen independently and at random. The square is then rotated 90^o clockwise about its center, and white square in a position formerly occupied by a black square is painted black. The colors of all other square are left unchanged. What is the probability the grid is now entirely black?

(A) $\frac{49}{512}$ (B) $\frac{7}{64}$

(C) $\frac{81}{512}$ (D) $\frac{9}{32}$

@ [A]

First, there is only one way for the middle square to be black because it is not affected by the rotation. Then we can consider the corners and edges separately. Let's consider the number of ways we can color the corners. There is 1 case with all black square. There are four cases with one white square and all work. There are six cases with white squares, but only the 2 with the white squares diagonal from each other work. There are no cases with three white squares of four white squares. Then the total number of ways to color the corners is 1 + 4 + 2 = 7. In essence, the edges work the same way, so there are also 7 ways to color them. The number of ways to fit the conditions over the number of ways to color the squares is

$$\frac{7 \times 7}{2^9} = \frac{49}{512}$$

25. At an elementary school, the students in third grade, fourth grade, and fifth grade run an average of 12, 15, and 10 minutes per day, respectively. There are twice as many third graders as fourth grades and twice as many fourth graders as fifth graders. What the average number of minutes run per day by these students?

(A) 12 (B) $\frac{37}{3}$

(C) $\frac{88}{7}$ (D) 14

@ [C]

Let there be x fifth graders. It follows that there are 2x fourth graders and 4x third graders. We have

$$\frac{(1x)(10)+(2x)(15)+(4x)(12)}{1x+2x+4x} = \frac{88}{7}$$

26. Set has 20 elements, and set B has 15 elements. What is the smallest possible number of elements in $A \cup B$?

(A) 5 (B) 15

(C) 20 (D) 35

@ [C]

$A \cup B$ will be smallest if b is completely contained in A, in which case all the elements in b would be counted for in A. So the total would be the number of elements in A, which is 20.

27. Last summer 30% of the birds living on Town Lake were geese, 25% were swans, 10% were herons, and 35 % were ducks. What percent of the birds that were not swans were geese?

(A) 20% (B) 30%

(C) 40% (D) 50%

@ [C]

75% of the total birds were not swans. Out of that was 30%/75% = 40% of the birds that were not swans that were geese.

28. Square EFGH has one vertex on each side of square ABCD. Point E is on AB with AE = 7 · EB. What is the ratio of the area of EFGH to the area of ABCD?

(A) $\frac{49}{64}$ (B) $\frac{25}{32}$

(C) $\frac{7}{8}$ (D) $\frac{5\sqrt{2}}{8}$

@ [B]

Let be the length of the sides of square ABCD, then the length of one of the sides of square EFGH is $\sqrt{(7s)^2 + s^2} = \sqrt{50x^2}$, and hence the ratio in the areas is $\frac{\sqrt{50s^2}^2}{(8s)^2} = \frac{50}{64} = \frac{25}{32}$

29. Two distinct regular tetrahedraL have all their vertices among the vertices of the same unit cube. What is the volume of the region formed by the intersection of the tetrahedraL?

(A) $\frac{1}{12}$ (B) $\frac{\sqrt{2}}{12}$

(C) $\frac{\sqrt{3}}{12}$ (D) $\frac{1}{6}$

@ [D]

A regular unit tetrahedron can be split into eight tetrahedra that have lengths of $\frac{1}{2}$. The volume of a regular tetrahedron can be found using base area and height.

For a tetrahedron of side length 1, Its base area is $\frac{\sqrt{3}}{4}$, and its height can be found using Pythagoras' Theorem. Its height is

$\sqrt{1^2 - \left(\frac{\sqrt{3}}{3}\right)^2} = \frac{\sqrt{2}}{\sqrt{3}}$ Its volume is $\frac{1}{3} \times \frac{\sqrt{3}}{4} \times \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2}}{12}$.

The tetrahedron actually has side length $\sqrt{2}$, so the actual volume is $\frac{\sqrt{2}}{12} \times \sqrt{2}^3 = \frac{1}{3}$

On the eight small tetrahedra. The four tetrahedra on the comers of the large tetrahdra are not inside the other tetrahedra. Thus. $\frac{4}{8} = \frac{1}{2}$ of the large tetrahedra will not be inside the other large tetrahedra.

The intersection of the two tetrahedra is thus $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

30. For which value of k does the following pair of equations yield a unique solution of x such that the solution is positive?

$x^2 - y^2 = 0$

$(x - k)^2 + y^2 = 1$

(A) 2 (B) 0

(C) $\sqrt{2}$ (D) $-\sqrt{2}$

@ [C]